(1+1)-dimensional massive sine-Gordon field theory and the Gaussian wave-functional approach

Wen-Fa Lu*
CCAST (World Laboratory), P.O. Box 8730, Beijing, 100080
and Department of Applied Physics, Shanghai Jiao Tong University, Shanghai 200030, China†
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The ground, one- and two-particle states of (1+1)-dimensional massive sine-Gordon field theory are investigated within the framework of the Gaussian wave-functional approach. We demonstrate that for a certain region of model parameter space, the vacuum of the field system is asymmetrical. Furthermore, it is shown that a two-particle bound state can exist upon the asymmetric vacuum for a part of the aforementioned region. In addition, for the bosonic equivalent to the massive Schwinger model, the masses of the one-boson and two-boson bound states agree with the recent second-order results of a fermion-mass perturbation calculation when the fermion mass is small. [S0556-2821(99)02010-X]

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I. INTRODUCTION

Massive sine-Gordon field theory (MSGFT) [1] is a simple generalization of massless sine-Gordon field theory (SGFT) [2], with a vacuum angle θ added in the argument of the cosine and a mass term $m_0^2\phi^2$ added in the Lagrangian. It is well known that SGFT is exactly solvable, and can provide a good laboratory for quantum field theory. Moreover, (1+1)-dimensional [(1+1)D] SGFT is equivalent to the massive $O(2)$ nonlinear σ model, the massive Thirring model, the two-dimensional Coulomb gas, and the continuum limit of the lattice x-y-z spin-$\frac{1}{2}$ model. Now this theory has received extensive investigations [3–5]. In the same way, MSGFT is also an important model. At any or some special coupling strength, MSGFT can give a good description for the dynamics of other important systems, such as the massive Schwinger model, the Schwinger-Thirring model, the two-dimensional lattice Abelian Higgs model, the two-dimensional neutral Yukawa gas, and so on [6–9]. And again, although it is not yet exactly solved owing to the existence of the mass term, this model possesses its own field-theoretical peculiarities [1], some of which will be discussed in this paper. Hence it is of general importance to study MSGFT. Early in the 1970s, this theory was analyzed within the framework of constructive quantum field theory [1]. Up until now, as an equivalent system of the massive Schwinger model (in this case, the coupling in MSGFT is only at a special strength), the MSGFT has been investigated for large m_0^2 by mass perturbation or some light-cone quantization methods [6,10–12]. In order to reveal the phase structure of the Abelian Higgs model, MSGFT with a finite momentum cutoff was treated by the renormalization-group technique [8] (1994). Obviously, further investigation of MSGFT (especially at any finite value of the coupling) is still necessary and of universal usefulness.

In this paper, using the Gaussian wave-functional approach (GWFA), we intend to investigate (1+1)D MSGFT with a zero vacuum angle $\theta=0$ at any coupling strength. The Lagrangian is

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x-\frac{1}{2}m_0^2\phi_x^2-\frac{m^2}{\beta^2}[1-\cos(\beta\phi_x)]$$
$$\equiv\frac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x-U(\phi_x), \quad (1)$$

with $\phi_x\equiv\phi(x)$, where m_0 and m are in mass dimension and the dimensionless β is the coupling parameter. It is always viable to have $\beta^2\geqslant 0$ [2]. In the case of $m_0=0$, Eq. (1) describes SGFT, and when $\beta^2\to 0$, Eq. (1) describes a free theory of the squared mass $(m_0^2+m^2)$ (if $m_0^2+m^2>0$). Evidently, the Lagrangian (1) is invariant under the transformation of $\phi\to(-\phi)$. We shall be particularly interested in spontaneous symmetry breakdown (SSB) and the two-particle bound states upon the SSB vacuum.[1] (In this paper, by SSB, we mean that the energy at a symmetric vacuum with $\phi=0$ is exactly higher than at an asymmetric vacuum with $\phi\neq 0$.) Also, we shall compare our results about the masses of the one-particle and the two-particle bound states with the ones in the literature.

We hope to demonstrate qualitatively the existence of SSB in MSGFT. As is known, the classical potential of SGFT is invariant under the transformation of $\phi\to(\phi+2n\pi/\beta)$ (hereafter n is an integer). Hence the classical vacua of SGFT are infinitely degenerate, and the corresponding quantized vacua are degenerate likewise [3,13]. But for MSGFT, the situation is quite different. A simple analysis

[1]Generally, symmetry breakdown also includes the phenomenon that the energy at a symmetric vacuum with $\phi=0$ is exactly equal to the one at an asymmetric vacuum with $\phi\neq 0$. This phenomenon is called degeneration in this paper.

*Email address: wenfalu@online.sh.cn
†Mailing address.

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indicates that because of the existence of the mass term, the classical vacuum of MSGFT is unique at $\phi=0$ for a negative m^2 with $|m^2|<m_0^2$ or for a positive m^2, whereas it is located at $\phi\neq 0$ for a negative m^2 with $|m^2|>m_0^2$. (In the case of $\beta^2=4\pi$, this is compatible with Ref. [12]. Note that in the caption of Fig. 5 of Ref. [12], the word ''large'' should perhaps be the words with the meaning ''sufficiently negative,'' according to the context there.) As suggested in Ref. [12], SSB is usually believed to be kept by the corresponding quantized vacuum. Undoubtedly, this phenomenon is interesting and useful both for electroweak theory and for the above equivalent models. Nevertheless, the investigations of MSGFT in the literature were mostly achieved for a small m^2 and made no explicit investigations of quantum SSB (to our knowledge). In the next section, we shall demonstrate the existence of SSB and give its relevant region in the model parameter space.

There exists a large variety of bound state systems in nature, but investigation of them is a hard task in quantum field theory [14]. For this task, the GWFA is an effective and feasible tool in practice and can give qualitatively correct results about bound states. Up to now, within the framework of the GWFA, a bound state has been shown to exist upon the symmetric vacuum of the following models: the (1+1)D :$\lambda(\phi^6-\phi^4)$: model [15], the ϕ^6 theory in (1+1) and (2+1) dimensions [16], the Gross-Neveu model [17], SGFT, and the double sine-Gordon model [13,18]. However, the GWFA has not established bound states upon the SSB vacuum of any model as yet. Although the $\lambda\phi^6$ theory has typically the SSB phenomenon, the interaction between the two particles is repulsive in this case and therefore the two particles form impossibly a bound state upon the SSB vacuum [16]. Perhaps MSGFT can give an example of such a phenomenon. Section III will concentrate on it and one will see that the phenomenon really exists in NSGFT.

In the last decade, the GWFA [19] has become a powerful tool to extract the nonperturbative information of many field theoretical models [15–17,20]. To be true, there have two unfavorable facts for the GWFA. One is that the GWFA gives the wrong order of the phase transition in $\lambda\phi^4$ theory [21,23], and the other is that it is difficult to control the approximate accuracy of the GWFA. Nevertheless, endeavors in the last decade have led to a little progress in controlling the accuracy. [22]. Moreover, the GWFA predicts correctly the existence of the phase transition [23,24] which may have second-order features [23,22], albeit it wrongly predicts the first order of the transition for some (1+1)D quantum field theories. A great deal of the existing work has shown that the GWFA is a tractable and helpful nonperturbative tool. It can give a qualitatively correct information [21] or a precursory study at least [25,22]. In the present work, we hope to give further helpful support to the GWFA through comparing our results with those obtained from the massive Schwinger model. As mentioned above, the system (1) at a special coupling strength ($\beta^2=4\pi$) is equivalent to the massive Schwinger model at the zero charge sector, in which there are only the Schwinger boson (fermion-antifermion bound state) and its bound states [6] (Coleman). Hence, the MSG bosons at $\beta^2=4\pi$ are just Schwinger bosons. Over the last two decades, the masses of the Schwinger boson and its bound states have been calculated at a good accuracy with the aid of many methods [11,26–30], most of which were numerically or analytically completed in the limit of the strong and/or weak coupling but not at all values of the coupling. On the other hand, our results about MSGFT are analytical at any set of values of the model parameters. Hence a comparison between ours and the others may be made. This will be done mainly for small fermion mass in Sec. IV. One will see that when $\beta^2=4\pi$, for the large m_0^2, i.e., for a small fermion mass parameter, the GWFA results for the masses of the Schwinger boson and its bound state have good agreement with those in Refs. [26–30].

In the next section, we shall directly give the Gaussian effective potential (GEP) of MSGFT and then discuss vacuum structure. As for the procedure of the GWFA, there have been detailed discussions in many references, such as Refs. [4,19–21,31], and we intend to give no more explanations in this paper. Section III is devoted to the excited states. For both the symmetric and the asymmetric vacua in Sec. II, we shall analyze existence of two-MSG-boson bound states and calculate the masses of the one-MSG-boson and two-MSG-boson bound states. In Sec. IV, the masses of the one-Schwinger-boson and two-Schwinger-boson bound state in the massive Schwinger model will be given from the results of MSGFT by employing the equivalence between the two models and compared with those in Refs. [29,30]. A brief conclusion and some discussions will be made at the end of this paper.

II. VACUUM STRUCTURE AND STABILITY

This section considers the ground state in the system (1).

In Eq. (1), we have to maintain a positive m_0^2 for avoiding an unbounded-below vacuum. Nevertheless, different from SGFT, both positive and the negative m^2 should be considered in Eq. (1) because the physics of the negative is not equivalent to the one of the positive. Moreover, as stated in the last section, the classical vacuum of the system (1) is infinitely degenerate no longer. It is symmetrical for a negative m^2 with $|m^2|<m_0^2$ or positive m^2, and becomes asymmetrical when m^2 is negative enough. In this section, we intend to investigate the structure and properties of the quantum vacuum through the GEP.

In the fixed-time functional Schrödinger picture, the normal-ordered Hamiltonian operator corresponding to Eq. (1) is

$$\mathcal{N}_M[H]=\int_x\left[\frac{1}{2}\Pi_x^2+\frac{1}{2}(\partial_x\phi_x)^2+\frac{1}{2}m_0^2\phi_x^2-\frac{1}{4}m_0^2I_1(M^2)-\frac{1}{2}I_0(M^2)+\frac{1}{4}M^2I_1(M^2)-\frac{m^2}{\beta^2}\mathcal{N}_M[\cos(\beta\phi_x)]\exp\left\{-\frac{\beta^2}{4}I_1(M^2)\right\}+\frac{m^2}{\beta^2}\right],\tag{2}$$

with the notation

$$I_n(M^2)=\int \frac{dp}{2\pi}\frac{\sqrt{p^2+M^2}}{(p^2+M^2)^n}.$$

Here, $\Pi_x \equiv -i\,\delta/\delta\phi_x$ is conjugate to the field operator ϕ_x, $\int_x \equiv \int dx$ the integration in one-dimensional coordinate space, and $\mathcal{N}_M[\cdots]$ means normal ordering with respect to any positive constant M (M is with mass dimension and usually called the normal-ordering mass). Take as an ansatz the general Gaussian wave functional [4,19,31]

$$|\Psi\rangle \to \Psi[\phi;\varphi,\mathcal{P},f]=N_f \exp\left\{ i\int_x \mathcal{P}_x\phi_x-\frac{1}{2}\int_{x,y}(\phi_x-\varphi_x)\times f_{xy}(\phi_y-\varphi_y)\right\}, \tag{3}$$

with N_f some normalization factor, and $\mathcal{P}_x$, φ_x as well as f_{xy} being the variational parameter functions. Using functional integration techniques [32,4,33], one can first calculate the energy $\int_x \langle\Psi|\mathcal{N}[\mathcal{H}_x]|\Psi\rangle$, then take φ_x as a constant φ, and finally minimize variationally the energy in respect to $\mathcal{P}$ as well as *f*. Consequently, $\mathcal{P}_x=0$, the Fourier component of f_{xy} is

$$f(p)=\sqrt{p^2+\mu^2(\varphi)}$$

and the GEP reads

$$V(\varphi)=\frac{1}{2}[I_0(\mu^2)-I_0(M^2)]-\frac{1}{4}[\mu^2 I_1(\mu^2)-M^2 I_1(M^2)] +\frac{1}{4}m_0^2[I_1(\mu^2)-I_1(M^2)]+\frac{1}{2}m_0^2\varphi^2 +\frac{m^2}{\beta^2}\left[1-\exp\left\{-\frac{\beta^2}{4}[I_1(\mu^2)-I_1(M^2)]\right\}\cos(\beta\varphi)\right]. \tag{4}$$

Here, μ takes one of the following three possible values: the nonzero root of the gap equation

$$\mu^2=\mu^2(\varphi)=m_0^2+m^2 \times \exp\left\{-\frac{\beta^2}{4}[I_1(\mu^2)-I_1(M^2)]\right\}\cos(\beta\varphi), \tag{5}$$

and the two end points of the range $0\leqslant\mu^2<\infty$, $\mu^2=0$ and $\mu^2\to\infty$ (the explanation of this point is put off to the next paragraph). In the right-hand side of Eq. (5), μ is a function of the uniform background field φ, which is the vacuum expectation of the field operator ϕ. Among the minimized results, $\mathcal{P}_x$ is the average value of the total momentum density operator of the field system, and its null result is understandable. As for $f(p)$ and μ, its physical meaning will get transparent in the next section. [By the way, because $U(\phi)$ has its Fourier representation in a sense of the tempered distribution [34], the above GEP and the energies of the one- and two-particle states in the next section can be also calculated directly as per formulas in Ref. [31], for which it is enough to finish some simple integrals.]

Noticing the results of the integrals,

$$\frac{1}{2}[I_1(\mu^2)-I_1(M^2)]=-\frac{1}{4\pi}\ln\frac{\mu^2}{M^2}$$

and

$$\frac{1}{2}[I_0(\mu^2)-I_0(M^2)]-\frac{1}{4}[\mu^2 I_1(\mu^2)-M^2 I_1(M^2)] =\frac{\mu^2-M^2}{8\pi},$$

one can see that Eqs. (4) and (5) contain no divergences, and therefore, a further renormalization procedure is not necessary. Nevertheless, in order to compute the GEP from Eq. (4), we have to choose the value of μ among the three possible values: 0, ∞, and the nonzero root of Eq. (5). The existence of the three possible values is because $\mu(\varphi)$ in Eq. (5) results from the process of minimizing the energy with respect to f [$\mu(\varphi)$ in Eq. (5) is the stationary point] and the GEP must be the global minimum of the energy density for the whole range of μ^2 $(0\leqslant\mu^2<\infty)$ [16,21,24]. Therefore, for every value of φ, one has to compare $\mathcal{V}(\varphi)$'s at the three possible values of μ with each other, and only the minimum among them can be taken as the GEP.

Obviously, the end point $\mu=0$ forces $V(\varphi)$ in Eq. (4) to be infinite and must be discarded. Moreover, for the other end point $\mu\to\infty$, one has

$$V(\varphi)\to\frac{\mu^2}{8\pi}-\frac{m^2}{\beta^2}\left(\frac{\mu^2}{M^2}\right)^{\beta^2/8\pi}\cos(\beta\varphi).$$

This implies that when $\beta^2<8\pi$, $V(\varphi)\to\mu^2/8\pi$ and tends to infinity for infinite μ. Thus, when $\beta^2<8\pi$, one should resort to only the nonzero solution of Eq. (5) for governing the GEP, which renders $V(\varphi)$ finite. As for the case of $\beta^2>8\pi$, we have

$$V(\varphi)\to-\frac{m^2}{\beta^2}\left(\frac{\mu^2}{M^2}\right)^{\beta^2/8\pi}\cos(\beta\varphi).$$

For those values of φ with $m^2\cos(\beta\varphi)>0$, the end point $\mu\to\infty$ makes $V(\varphi)$ unbounded from below, and accordingly the vacuum is unstable. So β^2 should be smaller than 8π. This constraint of β^2 is consistent with that in Ref. [1] (the seventh paragraph on p. 372 in the book), and is a little similar to that in SGFT [2,4] (the possible difference about the physical sense of the constraint will be discussed in Sec. V). In a word, for computing the GEP, we should use the nonzero root of Eq. (5) instead of the values $\mu^2=0$ and $\mu^2\to\infty$, and meanwhile, the coupling parameter β^2 is constrained to the range of $0\leqslant\beta^2<8\pi$. (For the SG systems in condensed matter physics, the constraint can be extended to $\beta^2<16\pi$ [35]. Perhaps, the constraint $\beta^2<8\pi$ could have some analogous extension for MSG systems in condensed matter physics.[2])

Furthermore, in order to analyze vacuum structure and stability, we still need the extremum condition $[dV(\varphi)/d\varphi=0]$

$$m_0^2\varphi+\frac{m^2}{\beta}\exp\left\{-\frac{\beta^2}{4}[I_1(\mu^2)-I_1(M^2)]\right\}\sin(\beta\varphi)=0 \tag{6}$$

and the stability condition (i.e., the second derivative of $\int_x\langle\Psi|\mathcal{N}[\mathcal{H}_x]|\Psi\rangle$ with respect to the relevant variational parameter f must be positive [4,21,24] at $\mu(\varphi)$)

$$1-\frac{m^2\beta^2}{8}\exp\left\{-\frac{\beta^2}{4}[I_1(\mu^2)-I_1(M^2)]\right\}I_2(\mu^2)\cos(\beta\varphi)>0. \tag{7}$$

When $\beta^2<8\pi$, Eq. (5) always has a nonzero solution (which is different from SGFT, where no solutions can exist for some values of $\beta\varphi$ [4]), and accordingly we can define a parameter with mass dimension

$$\mu_0^2\equiv\mu^2(\varphi=0)=m_0^2+m^2\exp\left\{-\frac{\beta^2}{4}[I_1(\mu_0^2)-I_1(M^2)]\right\}, \tag{8}$$

which is positive (independent of the sign of m^2), and is physical mass squared when the vacuum is symmetric (see the next section). When $m^2<0$, $\mu_0<m_0$; otherwise, $\mu_0>m_0$. In Eqs. (4), (5), (6), and inequality (7), we can eliminate m^2 in favor of μ_0^2 with the help of the definition (8). For convenience of numerical computation, we define the following dimensionless quantities:

$$\tilde{V}(\varphi)\equiv\frac{V(\varphi)-V(\varphi=0)}{\mu_0^2},\quad \tilde{\mu}\equiv\frac{\mu}{\mu_0},\quad \tilde{m}_0\equiv\frac{m_0}{\mu_0}. \tag{9}$$

Then, we can rewrite the GEP as

$$\tilde{V}(\varphi)=\left(\frac{1}{8\pi}-\frac{1}{\beta^2}\right)(\tilde{\mu}^2-1)-\frac{\tilde{m}_0^2}{8\pi}\ln(\tilde{\mu}^2)+\frac{1}{2}\tilde{m}_0^2\varphi^2, \tag{10}$$

the gap

$$\tilde{\mu}^2=\tilde{m}_0^2+(1-\tilde{m}_0^2)(\tilde{\mu}^2)^{\beta^2/8\pi}\cos(\beta\varphi), \tag{11}$$

the extremum condition

$$\tilde{m}_0^2\varphi+\frac{\tilde{\mu}^2-\tilde{m}_0^2}{\beta}\tan(\beta\varphi)=0, \tag{12}$$

and the stability condition

$$1-\beta^2\frac{\tilde{\mu}^2-\tilde{m}_0^2}{8\pi\tilde{\mu}^2}>0. \tag{13}$$

Thus, the calculation of the GEP from the last four equations or inequality is equivalently carried out at a fixed value of m^2. When one intends to consider the effects of m^2, it is enough to further utilize the original definition Eqs. (8) and (9). (Of course, in order to define dimensionless quantities, one can have other choices. For example, one can choose the parameter m_0^2 as a unit instead of μ_0.)

Now we can compute the GEP of MSGFT. It is difficult to solve analytically Eqs. (10)–(12) and the inequality (13), and hence we have to appeal to a numerical method for tackling them. If there exists a SSB phenomenon, analysis of the vacuum structure indeed amounts to the determination of the boundary between the symmetric and asymmetric vacua in the model parameter space β^2-$\tilde{m}_0^2$. In order to obtain the boundary, one can search for the points in the space β^2-$\tilde{m}_0^2$ at each of which the value of the GEP for the asymmetric vacuum ($\varphi\neq0$) is exactly equal to that for the symmetric vacuum. When one executes the numerical computation, an additional point to be noticed is that for some values of the parameters $\{\beta^2,\tilde{m}_0^2\}$ there exist two roots of Eq. (11) and one of them should be chosen so as to minimize $\tilde{V}(\varphi)$ in Eq. (10).

The numerical results indicate that (i) when $\tilde{m}_0^2<2$ (an approximate value), the vacuum is symmetrical; (ii) for any $\tilde{m}_0^2>2$, there is a critical value of the coupling parameter β_c^2, at which the vacuum is degenerate and located at either $\varphi=0$ or $\varphi\neq0$. When $\beta^2<\beta_c^2$ the vacuum is symmetrical, whereas when $\beta^2>\beta_c^2$, the symmetry of the vacuum is broken. Collecting all the above, we depict the $\tilde{m}_0^2$-β^2 parameter space in Fig. 1. The allowed region of the parameters at any fixed μ_0 forms a semi-infinite strip $\{\tilde{m}_0^2>0, 0\leqslant\beta^2<8\pi\}$, and in this strip, the long-dashed curve represents the critical coupling β_c^2. (The dotted and short-dashed curve are relevant to bound states and will be explained in the next section.) In Fig. 1, domain I corresponds to the symmetric vacuum, and domains II and III correspond to the asymmetric vacuum. From this figure, one can see that with the increase of $\tilde{m}_0$ the domain I gets more and more narrow. That is to say, the more negative m^2 is, the wider the domain of β^2 for the asymmetric vacuum is. For an illustrative purpose, we plot the GEP in Figs. 2 and 3 for $\tilde{m}_0^2=1.5$ and $\tilde{m}_0^2=20$, respectively. For the latter, $\beta_c^2\approx3.0265772$. (Note that the GWFA value of β_c^2 could not predict an exact value of the critical point when a phase transition near β_c^2 is considered, and perhaps the value of β_c^2 and some relevant information could be changed by some better approximate approach. The discussion relevant to this point will be deferred to Sec. V.)

[2]To determine this point needs some further investigation.

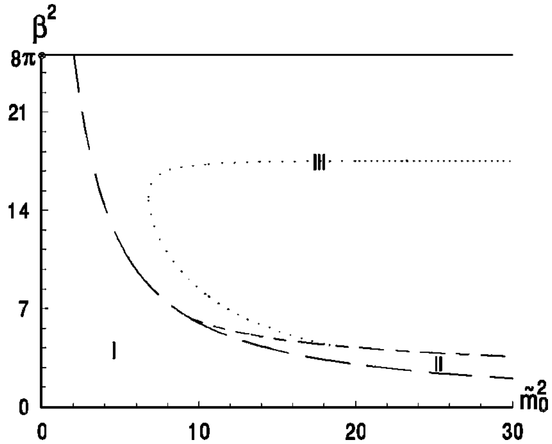

FIG. 1. The β^2-$\tilde{m}_0^2$ parameter space for the massive sine-Gordon field theory in 1+1 dimensions, which is plotted from Eqs. (10)–(13). The vacuum is symmetric in domain I, and is asymmetric both in domain II without a bound state and in domain III with a bound state. The short-dashed curve corresponds to $\cos(\beta\varphi_0)=0$ and the dot-dashed curve to the vacuum at $\varphi_0=0.75$.

Thus, we see that when $\tilde{m}_0^2>2$, there really exists an asymmetric vacuum within the framework of the GWFA. That is to say, for MSGFT, the classical double-well potential can cause a SSB in quantum theory. This is usually believed, just as pointed out in Ref. [12]. When $\beta^2<16/\pi$, this is also compatible with Ref. [1] (p. 382).

About the asymmetrical vacuum of MSGFT, we have more to say. Fröhlich [1] pointed out that for m^2 large enough and positive, $\phi\to(-\phi)$ symmetry is presumably dynamically broken. This implies that for a sufficiently small $\tilde{m}_0^2$, the vacuum can be asymmetrical. Nevertheless, using the above GWFA results, we failed to find a very small but nonzero value of $\tilde{m}_0^2$ (with $\beta^2<8\pi$) which can lead to a dynamic symmetry breakdown. (In order to consider it, we also chose m_0^2 as a unit to perform the numerical calculation

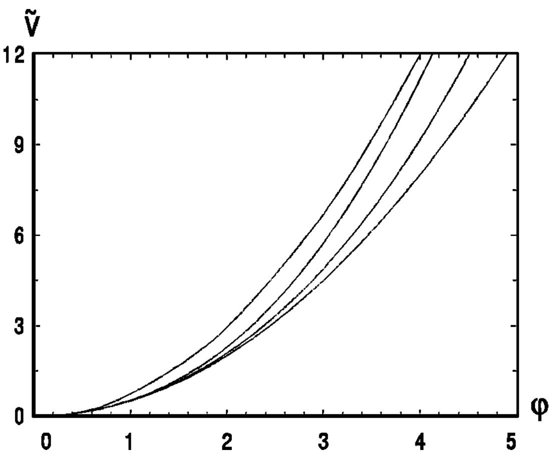

FIG. 2. The reduced GEP of the massive sine-Gordon field theory in 1+1 dimensions for the case of $\tilde{m}_0^2=1.5$. Only one-half of the symmetric potential is shown. In this figure, curves from the left to the right are drawn for $\beta^2=0.0004$, 0.25, 1, and 25, respectively.

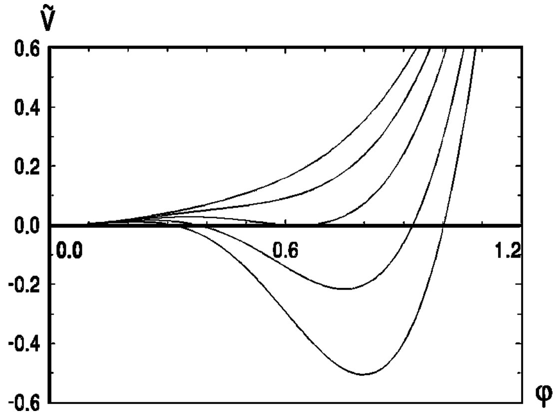

FIG. 3. Similar to Fig. 2 but in $\tilde{m}_0^2=20$. In this figure, curves from the highest to the lowest are drawn for $\beta^2=1.44$, 2.25, $\beta_c\approx3.03$, 4.0, and 4.84, respectively.

and failed likewise.) Of course, when $\tilde{m}_0^2$ tends to zero, the GWFA can well give the effective potential for $\beta\varphi\in[(2n-\frac{1}{2})\pi,(2n+\frac{1}{2})\pi]$ (in the case of $m^2>0$) [4], and the vacua are degenerate, which is a special case of dynamical symmetry breakdown. Thus, in Fig. 1, the left, linear boundary of the domain I ($\tilde{m}_0^2=0$, $0\geqslant\beta^2<8\pi$) corresponds to the special dynamical-symmetry-breakdown vacuum, which can be either symmetrical or asymmetrical at each point of the boundary.

Additionally, we want to mention the symmetry restoration by quantum effects [3,36,24]. From Eqs. (8) and (9), $\tilde{m}_0^2>1$ means $m^2<0$, and thus the above symmetry-vacuum domain with $\tilde{m}_0^2>1$ in Fig. 1 shows the occurrence of a symmetry restoration phenomenon, because the corresponding classical vacuum can be spontaneously symmetry broken.

In this section, we have obtained the vacuum structure of MSGFT. The ground state wave functional is the best trial Gaussian functional $\Psi[\phi;\varphi,\mathcal{P},f]$ in Eq. (3) with $\mathcal{P}_x=0$, $\varphi_x=\varphi_0$, and the Fourier component of f_{xy} is $f(p)=\sqrt{p^2+\mu^2(\varphi_0)}$. Here φ_0 is a constant satisfied by Eq. (6), at which the GEP is lowest and the vacuum is located. In addition, there is a constraint of β^2, that is, $0\leqslant\beta^2<8\pi$. In the next section, we shall discuss the excited states upon the ground state. For convenience, we use $|\varphi_0\rangle$ to represent the ground state wave functional hereafter.

III. BOUND STATES

In this section, we investigate the one- and two-particle excited states. Following Refs. [15,16,19,31,33], one can manufacture the annihilation and creation operators with respect to the vacuum state $|\varphi_0\rangle$:

$$A_f(p)=\left(\frac{1}{4\pi f(p)}\right)^{1/2}\int_x e^{-ipx}\left[f(p)(\phi_x-\varphi_0)+\frac{\delta}{\delta\phi_x}\right] \tag{14}$$

and

$$A_f^\dagger(p)=\left(\frac{1}{4\pi f(p)}\right)^{1/2}\int_x e^{ipx}\left[f(p)(\phi_x-\varphi_0)-\frac{\delta}{\delta\phi_x}\right]. \tag{15}$$

It is evident that $A_f(p)|\varphi_0\rangle=0$ and the commutator $[A_f(p),A_f^\dagger(p')]=\delta(p-p')$. Then one has the one-particle state

$$|1\rangle=A_f^\dagger(p)|\varphi_0\rangle \tag{16}$$

and the *S*-wave two-particle state

$$|2\rangle=\int dp\,\Sigma(p)A_f^\dagger(p)A_f^\dagger(-p)|\varphi_0\rangle, \tag{17}$$

where $\Sigma(p)$ is the Fourier transformation of the *S*-wave function of the two-particle system.

For the one-particle state, one can find

$$m_1=\frac{\langle 1|\mathcal{N}_M[H]|1\rangle}{\langle 1|1\rangle}-\int_x \mathcal{V}(\varphi_0)=\sqrt{p^2+\mu^2(\varphi_0)}, \tag{18}$$

which is the energy of one particle with a momentum p. This is the physical sense of $f(p)$, which has previously appeared in the last section. Obviously, $\mu(\varphi_0)$ is m_R, the physical mass of a particle according to the relevant vacuum. Thus, within the framework of the GWFA, the single-particle mass of MSGFT is

$$m_R^2=m_0^2+m^2\left(\frac{m_R^2}{M^2}\right)^{\beta^2/8\pi}\cos(\beta\varphi_0). \tag{19}$$

A further analysis tells us that for both the asymmetric and the symmetric vacua, m_R increases with the increase of m_0^2 or m^2; for the symmetric vacuum, m_R also increases when β^2 increases, but for the asymmetric vacuum, things get a little complicated, which here we intend to discuss no longer.

Now we turn to discuss the two-particle state. From Ref. [33], the two-particle energy $m_2=\langle 2|H|2\rangle/\langle 2|2\rangle-\int_x\mathcal{V}(\varphi_0)$ can be calculated as (expressed in terms of the dimensionless quantities)

$$\tilde{m}_2=\frac{2\int d\tilde{p}[\Sigma(\tilde{p})]^2 f(\tilde{p})-(\beta^2/16\pi)[\tilde{\mu}^2(\varphi_0)-\tilde{m}_0^2]\left[\int d\tilde{p}\,\Sigma(\tilde{p})/f(\tilde{p})\right]^2}{\int d\tilde{p}[\Sigma(\tilde{p})]^2}, \tag{20}$$

with $\tilde{m}_2\equiv m_2/\mu_0$, $\tilde{p}\equiv p/\mu_0$, and $f(\tilde{p})\equiv\sqrt{\tilde{p}^2+\tilde{\mu}^2(\varphi_0)}$. The two terms in this expression can be regarded as the kinetic energy of the two constituent particles and their interacting energy, respectively. Obviously, the interacting energy is closely related to $[\tilde{m}_0^2-\tilde{\mu}^2(\varphi_0)]$. When $\tilde{\mu}^2(\varphi_0)<\tilde{m}_0^2$, the interacting energy is positive, and the two particles repel each other and cannot combine into a bound state, while for $\tilde{\mu}^2(\varphi_0)>\tilde{m}_0^2$, the interacting energy is negative, and the two particles attract each other and may form a bound state.

Analyzing Eq. (11), one can find that for symmetry vacuum ($\varphi_0=0$), i.e., for domain I in Fig. 1, when $\tilde{m}_0<1$, $\tilde{\mu}^2(\varphi_0)>\tilde{m}_0^2$, and when $\tilde{m}_0>1$, $\tilde{\mu}^2(\varphi_0)<\tilde{m}_0^2$; for a symmetry-broken vacuum ($\varphi_0\neq 0$), i.e., for domains II and III in Fig. 1, if $\cos(\beta\varphi_0)>0$, then $\tilde{\mu}^2(\varphi_0)<\tilde{m}_0^2$, and if $\cos(\beta\varphi_0)<0$, then $\tilde{\mu}^2(\varphi_0)>\tilde{m}_0^2$. It is worthwhile noticing that the case of $\cos(\beta\varphi_0)<0$ does exist when $\tilde{m}_0^2$ is less than about 9.5 or when β^2 is greater than some value β_b^2 for any larger $\tilde{m}_0^2$, which is involved in domain III in Fig. 1. In Fig. 1, the short-dashed curve corresponds to the critical case of $\cos(\beta_b\varphi_0)=0$. Thus, for the symmetry vacuum with $\tilde{m}_0>1$ or for the asymmetry vacuum with $\cos(\beta\varphi_0)>0$ (domain II) the two-particle states can be just the scattering ones, whereas for the symmetry vacuum with $\tilde{m}_0<1$ or for the asymmetry vacuum with $\cos(\beta\varphi_0)<0$ (domain III), there can exist the two-particle bound states.

The mass of the bound state m_b can be calculated within the framework of the GWFA [15,16,13]. Minimizing the energy $\tilde{m}_2$ with respect to $\Sigma(\tilde{p})$ yields the equation for $\tilde{m}_2$:

$$\frac{\beta^2}{16\pi}[\tilde{\mu}^2(\varphi_0)-\tilde{m}_0^2]\int\frac{d\tilde{p}}{f^2(\tilde{p})[2f(\tilde{p})-\tilde{m}_2]}=1. \tag{21}$$

When $\tilde{\mu}^2(\varphi_0)<\tilde{m}_0^2$ this equation has no solution but one can obtain the scattering phase shifts [33,5]. When $\tilde{\mu}^2(\varphi_0)>\tilde{m}_0^2$, Eq. (21) has a solution with $\tilde{m}_2<2\tilde{\mu}(\varphi_0)$, and $\tilde{m}_2$ in this equation times μ_0 is just the mass of the bound state m_b. Finishing the integration in Eq. (21) leads to

$$\tilde{m}_b=\frac{\beta^2}{8\pi}\left(1-\frac{\tilde{m}_0^2}{\tilde{\mu}^2(\varphi_0)}\right)\left[\frac{1}{\sqrt{1-\tilde{m}_b^2}}\tan^{-1}\sqrt{\frac{1+\tilde{m}_b^2}{1-\tilde{m}_b^2}}-\frac{\pi}{4}\right], \tag{22}$$

with the reduced mass $\tilde{m}_b\equiv m_b/2\mu(\varphi_0)$. When $\tilde{m}_0^2<1$, the vacuum is symmetric, $\tilde{\mu}(\varphi_0)$ is unity, and the last equation is enough to give the reduced mass of the bound state in the symmetric vacuum. For some values of $\tilde{m}_0^2$, we give the dependence of the reduced mass upon the coupling constant β^2 in Fig. 4. This figure indicates that the reduced mass of the bound state decreases with an increase of β^2, and increases

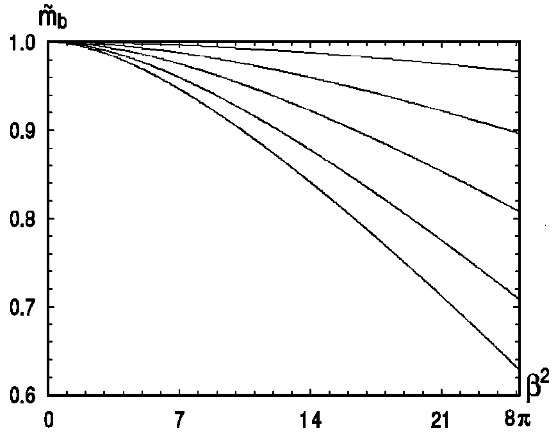

FIG. 4. The dependence of the reduced mass of the bound state in the symmetric vacuum upon β^2 for some values of $\tilde{m}_0^2$. In this figure, curves from the lowest to the highest are drawn for $\tilde{m}_0^2 = 0.05$, 0.2, 0.4, 0.6, and 0.8, respectively.

with an increase of $\tilde{m}_0^2$. [Note that for the symmetry vacuum case, Eq. (22) can give vanishing or even negative $\tilde{m}_b$ if the curves in Fig. 4 are not artificially cut off at $\beta^2 = 8\pi$.] For a given asymmetric vacuum, one can calculate the reduced mass of the bound state through Eqs. (10)–(13) and (22). In this case, a vacuum located at a different φ_0 corresponds to a different curve in domain III of Fig. 1. (Of course, so it does in domain II.) For instance, the vacuum at $\varphi_0 = 0.75$ corresponds to the dotted curve in domain III. Vacua at other φ_0 correspond to other similar curves. For $\varphi_0 = 0.75$, we depict the dependence of the reduced mass of bound state upon β^2 in Fig. 5 and upon $\tilde{m}_0^2$ in Fig. 6. From Fig. 5, the reduced mass of the bound state decreases with the increase of β^2, which is similar to that for the symmetric vacuum, but is almost fixed at some not-too-small value (0.76 or so) before β^2 arrives at the limit 8π. The dependence upon $\tilde{m}_0^2$ is slightly complex from Fig. 6. When two particles are not tightly bound, the reduced mass of the bound state increases with the increase of $\tilde{m}_0^2$, and approaches the value 1 (two-free-particle case) when $\tilde{m}_0^2$ rises at such a value that the

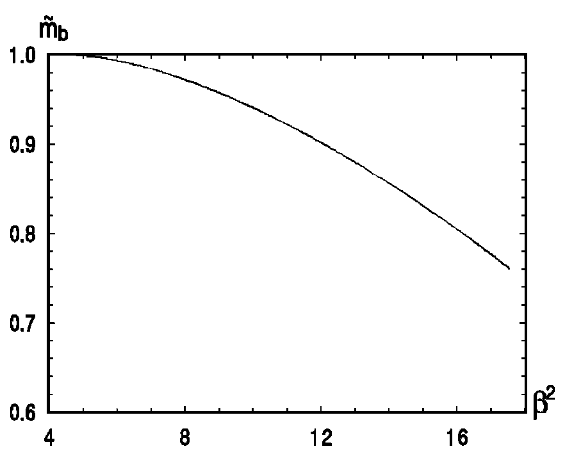

FIG. 5. The dependence of the reduced mass of the bound state in the asymmetric vacuum $\varphi_0 = 0.75$ upon β^2.

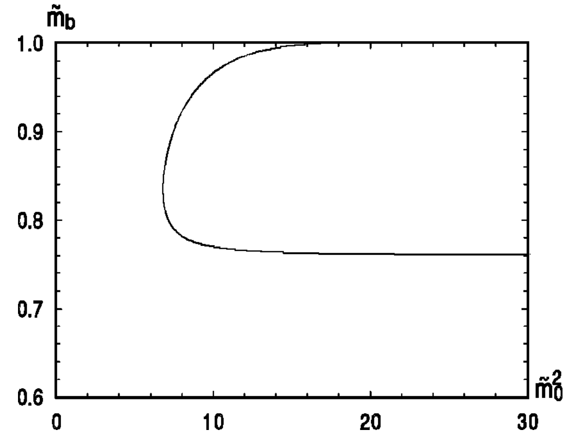

FIG. 6. The dependence of the reduced mass of the bound state in the asymmetric vacuum $\varphi_0 = 0.75$ upon $\tilde{m}_0^2$.

dotted curve in Fig. 1 falls down on the short-dashed curve, which means that $\tilde{m}_0^2$ acts as the coefficient of the free term in the Lagrangian. On the other hand, if the binding between two particles is slightly tighter, the reduced mass decreases with the increase of $\tilde{m}_0^2$, which means that $\tilde{m}_0^2$ acts as the coefficient of the cosine interaction term in the Lagrangian, and stays at the fixed value 0.76 or so for larger $\tilde{m}_0^2$. The monotonous decrease of $\tilde{m}_b$ upon $\tilde{m}_0^2$ is understandable because $(1 - \tilde{m}_0^2)$ plays the role of m^2/μ_0^2 in the reduced expression (11). These results indicate that from Eq. (22) the bound state in an asymmetric vacuum never becomes ultratightly bound.

In this section, we have obtained the single-particle mass of MSGFT, and shown that for both the symmetric and the asymmetric vacua, there exist two-particle bound states. Moreover, we have also given the bound-state mass. Next, we shall compare the above masses upon the symmetric vacuum with the ones in the literature.

IV. SCHWINGER BOSON AND ITS BOUND STATE

As pointed out in the Introduction, the (1+1)D MSGFT equation (1) with $\beta^2 = 4\pi$ is equivalent to the massive Schwinger model at the zero charge sector. The Lagrangian of the latter is [1,6,9,11,12,27] (Carroll *et al.*)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}_x [\gamma^\mu (i\partial_\mu - eA_\mu) - m_f] \psi_x, \quad (23)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The normal-ordering Hamiltonian corresponding to Eq. (23) with normal-ordering mass m_f is equal to Eq. (2) with the normal-ordering mass m_0. The correspondence between the parameters in Eqs. (1) and (23) is

$$m_0^2 = \frac{e^2}{\pi}, \quad m^2 = 2e^\gamma m_0 m_f, \quad (24)$$

where γ is the Euler constant. Equation (23) is (1+1)D QED with a massive fermion, and is obtained by inserting an additional mass term $m_f\bar{\psi}\psi$ in the Lagrangian of the Schwinger model [37]. The Schwinger model was exactly solved [37], and shares some nontrivial properties with QCD such as nontrivial vacuum, quark trapping, and so on. It is equivalent to a massive free-scalar-field theory with the boson mass m_0. For the particle spectrum of this model, there exist no single-fermion states but a free boson which is a fermion-antifermion bound state (usually called a Schwinger boson). Having an additional mass term, the massive Schwinger model has a Schwinger boson and its various excited states. In view of those aspects of the Schwinger model, most investigations of the massive Schwinger model were involved in tackling the confinement and particle spectrum of it, and mainly in small-m_f effects. To our knowledge, except for a lattice study (not including those on light cone), only Refs. [26,28] gave the masses of the Schwinger boson and its bound state for a finite m_f. In this section, we shall give the masses of the Schwinger boson and two-Schwinger-boson bound state from the symmetric-vacuum results in the last section, and compare them with the recent results in Refs. [29,30]. As for the vacuum structure, if m_f in Eq. (23) is greater than zero or m_f is small, there exists no asymmetric vacua according to the results in Sec. II [$m_f>0$ implies $m^2>0$, and hence $\tilde{m}_0^2<1$ from Eq. (8)].

We first consider the Schwinger boson. From Eq. (19), one can have the Schwinger boson mass m_s upon a symmetric vacuum ($\varphi_0=0$) (taking $M=m_0$)

$$\tilde{m}_s=e^{\gamma}\tilde{m}_f+\sqrt{e^{2\gamma}\tilde{m}_f^2+1}, \tag{25}$$

with $\tilde{m}_s=m_s/m_0$ and $\tilde{m}_f=m_f/m_0$. When $\tilde{m}_f$ is infinitesimal, we obtain, from the last equation,

$$\tilde{m}_s^2=1+2e^{\gamma}\tilde{m}_f+2e^{2\gamma}\tilde{m}_f^2+O(\tilde{m}_f^3). \tag{26}$$

Performing the fermion-mass perturbation technique for the massive Schwinger model in the ''near'' light-front coordinate system, Ref. [29] gave the Schwinger boson mass to second order of m_f. We find that the first two terms on the right-hand side of the last equation are identical to the corresponding terms in Eq. (3.16) of Ref. [29], and the only difference is that the constant factor in the m_f^2 term is $2e^{2\gamma}$ for our result but $e^{2\gamma}$ for Eq. (3.16) in Ref. [29]. More recently, Ref. [30] also gave almost the identical result of the Schwinger boson mass up to second order of m_f with that in Ref. [29]. Thus, for an infinitesimal m_f, our result of the Schwinger boson mass is in good agreement with the ones in Refs. [29,30]. For an illustration, a plot of our result and the results in Refs. [29,30] is shown in Fig. 7. In this figure, the results in Refs. [29,30] are represented by dashed curves and coincide with each other. This figure indicates that with the increase of $\tilde{m}_f$, our result (solid curve) is more and more higher than the dashed curve, while for small $\tilde{m}_f<0.2$, the solid curve nearly coincides with the dashed curve.

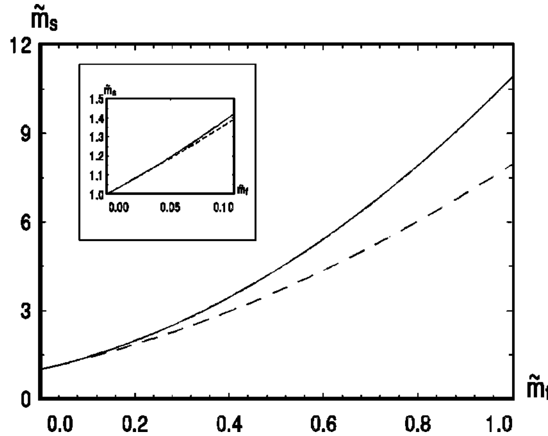

FIG. 7. The comparison of the Schwinger boson mass, Eq. (26) (solid curve), with the corresponding second-order results in Refs. [29,30] (dashed curve).

Now we are in a position to discuss the two-Schwinger-boson bound state. From Eq. (22), we gain the mass of the two-Schwinger-boson state upon the symmetric vacuum m_{sb}:

$$\tilde{m}_{sb}=\frac{\tilde{m}_f^2}{2\tilde{m}_s}\left[\frac{1}{\sqrt{1-\tilde{m}_{sb}^2}}\tan^{-1}\sqrt{\frac{1+\tilde{m}_{sb}^2}{1-\tilde{m}_{sb}^2}}-\frac{\pi}{4}\right], \tag{27}$$

with the reduced mass $\tilde{m}_{sb}\equiv m_{sb}/2m_s$. From Eqs. (26) and (27), we depict $\tilde{m}_{sb}$ with respect to the small reduced fermion mass $\tilde{m}_f$ in Fig. 8 (the solid curve). In this figure, the dashed curve is the corresponding result in Ref. [30] to second order of the fermion mass m_f. This figure demonstrates that for a small $\tilde{m}_f$ the result from the GWFA agrees very well with the second-order result of fermion-mass perturbation in Ref. [30].

According to Refs. [26–30], the mass-perturbation results are in good agreement with analytical or numerical ones

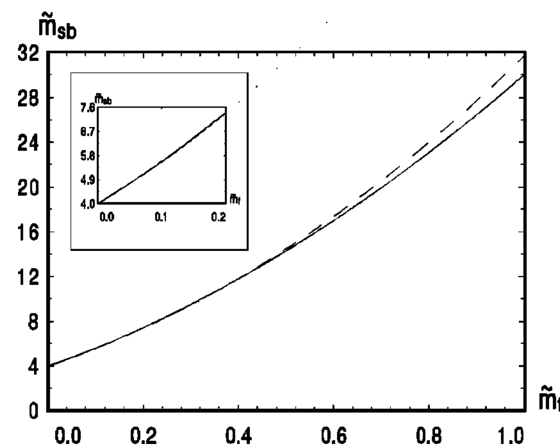

FIG. 8. The dependence of the reduced mass of the two-Schwinger-boson bound state in the symmetric vacuum $\tilde{m}_{bs}$ upon the reduced mass $\tilde{m}_f$. The dashed curve is the corresponding result in Ref. [30].

from many other techniques. Therefore, we can say that for a small mass m_f, the GWFA gives the masses of Schwinger boson and two Schwinger-boson bound state with good accuracy.

By the way, for finite fermion mass, the mass $\tilde{m}_s$ from Eq. (25) is nearly linear in terms of $\tilde{m}_f$. We notice that in Ref. [26], the result of the mass of the Schwinger boson is also a linear one in terms of the m_f. For any given value of m_f, there exist infinitely many results from Ref. [26]. For example, for $\tilde{m}_f = 2.0$, besides the results in Figs. 2 and 3, Ref. [26] gave the other four values of the Schwinger boson mass: 4.78, 5.97, 6.9, and 7.70. For this case $\tilde{m}_f = 2.0$, Eq. (25) gives $\tilde{m}_s = 7.262$, which is between the last two values 6.9 and 7.7. The relative error is 6% or so.

V. CONCLUSION

In this paper, we investigated MSGFT with the GWFA in (1+1) dimensions. We discussed the ground, one-, and two-particle states. For the ground state, we demonstrated the existence of the asymmetric vacuum, obtained the constraint of the coupling $\beta^2 < 8\pi$, and gave the parameter regions of the symmetric and the asymmetric vacua (Fig. 1). For the one-particle state, the implicit formula (19) is obtained for the mass of a single MSG particle. As for the two-particle state, we show that the two-particle bound state can exist upon an asymmetric vacuum. We also give the bound-state mass formula (22) as well as the model parameter regions of the bound states upon the symmetric and the asymmetric vacua, and discuss the dependence of the bound-state masses upon the model parameters β^2 and $\tilde{m}_0^2$. Finally, the masses of the Schwinger boson and the two-Schwinger-boson bound state in the massive Schwinger model are calculated according to Eqs. (19) and (22), and are in good agreement with those in the recent literature [29,30] (Figs. 7 and 8) when the fermion mass m_f is small.

Before closing the paper, we want to give a further discussion of Figs. 1 and 3. Figure 1 is just the phase diagram of the vacuum. In this figure, with continuous variations in β^2 and $\tilde{m}_0^2$, a symmetric phase of the vacuum can transit the long-dashed boundary to an asymmetric phase. This implies the occurrence of a phase transition. Furthermore, Fig. 3 is an explicit illustration of the GEP with $\tilde{m}_0^2 = 20$, and it indicates that the vacuum average value of the field operator φ_0 changes discontinuously from zero to nonzero when β^2 increases gradually. That is to say, the GWFA predicts a first-order phase transition in MSGFT. Nevertheless, it could be inadvisable to conclude that a true first-order phase transition occurs in MSGFT. As mentioned in Sec. II, some GWFA information related to the phase transition may be changed by a better approximation method. The GWFA is a simple nonperturbative approach. Although it is effective and useful for investigating many problems or phenomena, we should not expect too much of it, particularly when we are concerned with a phase transition. In fact, for a few (1+1)D field theories, the GWFA predicts the wrong order of the phase transition. We take the $\lambda\phi^4$ field theory as the first example. Simon and Griffiths gave a rigorous theorem that for the (1+1)D $\lambda\phi^4$ field theory in the presence of an external field $B \neq 0$, the vacuum of it is unique [38]. Further, Chang proved that the occurrence of the first-order phase transition in the (1+1)D $\lambda\phi^4$ field theory violates the Simon-Griffiths theorem, but a second-order phase transition can be compatible with this theorem [23]. As is known, the GWFA predicts just a first-order phase transition in this theory [23,21], and a second-order phase transition can occur in the (1+1)D $\lambda\phi^4$ field theory [23,39]. That is to say, the GWFA predicts correctly the existence of the phase transition in the (1+1)D $\lambda\phi^4$ field theory, and predicts incorrectly just the nature of the phase transition. Another example is the (1+1)D ϕ^6 field theory. The GWFA predicts a first-order phase transition in this theory [16]. But a coupled-cluster-method investigation indicated that for the region where the two-particle bound state exists, a first-order phase transition can occur in the (1+1)D ϕ^6 field theory; nevertheless, the critical curve is different from the corresponding one in the GWFA result [40]. Additionally, for the region where a two-particle bound state disappears, no first-order phase transitions exist, but a second-order phase transition is believed to occur in the (1+1)D ϕ^6 field theory [40]. In view of these situations of the above two theories, we feel that the Simon-Griffiths theorem could perhaps have an effect on the other (1+1)D field theories to some extent. Therefore, we conjecture that for (1+1)D MSGFT, the GWFA predicts correctly the existence of a phase transition, but could make a mistake in predicting the critical boundary and the nature of the transition, perhaps which will be similar to those in the (1+1)D ϕ^6 field theory. In order to get a definite conclusion, some better approximate methods should be used [22,39,40]. We believe that after considering the higher order correction of the GEP [22,39,40], one may obtain different figures from Figs. 1 and 3, but the asymmetric vacuum would still exist. In a general, when a classical vacuum in 1+1 dimensions is spontaneously symmetry broken for some region of the model parameters, quantum effects are not sufficient to make the vacuum completely symmetrical for all values of the model parameters. The existence of the asymmetric vacuum should be reasonable.

Besides, in Fig. 1, when $\tilde{m}_0^2$ tends to zero, the symmetric phase of the vacuum becomes a dynamical-symmetry-breakdown phase. This perhaps implies the existence of a phase transition. In fact, Ref. [1] (the end paragraph on p. 407 in the book) pointed out that there may be a phase transition (and long-range order) if m_0 is small enough. Additionally, as mentioned in Sec. II, the constraint of $\beta^2 < 8\pi$ in MSGFT is the same as in the SGFT. It is well known that $\beta^2 = 8\pi$ is a critical point at which the Kosterlitz-Thouless transition occurs in the SG system [41]. However, when $\tilde{m}_0^2 \neq 0$, $\beta^2 = 8\pi$ could not suggest that the Kosterlitz-Thouless transition and the zero-mass phase could not exist, for $\mu = 0$ cannot give either the local or the global minimum of the energy density but infinity (see Sec. II). As a matter of fact, the disappearance of the massless Kosterlitz-Thouless phase has been shown in Ref. [8] (1994). By the way, if one uses the GWFA with a finite momentum cutoff [42,18]

(Zhang *et al.*), it is possible to have a satisfactory understanding about the problems discussed in this paragraph.

In conclusion, the results in this paper are qualitatively correct and are necessary and helpful for further investigations of MSGFT at least. The GWFA results about the ground state of MSGFT are useful for the Yukawa gas and the lattice Abelian Higgs model, and meanwhile discussions about other properties or phenomena of the massive Schwinger model with the GWFA will be also interesting and useful.

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